De Morgan's Laws in Fuzzy Hypersoft set Theory

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Abstract

This study explores De Morgan's laws within the realm of fuzzy hypersoft set theory. Expanding upon the research conducted by Taha Yasin Öztürk and Adem Yolcu, it examines the principles of inclusion and equality, highlighting their relevance and underlying theoretical foundations. The findings contribute to a deeper understanding of fuzzy hypersoft set structures while fostering both theoretical advancements and practical implementations.

Keywords: De Morgan's laws, fuzzy hypersoft set.

1. Introduction

De Morgan's Laws serve as a cornerstone of classical set theory and algebra, offering a mathematical framework for analyzing the relationships between union, intersection, and complementation. Originally formulated by Augustus De Morgan in the 19th century, these laws provide fundamental principles for logical and set-theoretic operations.

In fuzzy set theory, De Morgan's Laws are extended to accommodate degrees of membership, where elements belong to sets with varying levels of certainty rather than absolute inclusion or exclusion. This concept becomes even more complex in fuzzy hypersoft set theory, an advanced generalization of fuzzy soft sets that incorporates multiple parameters influencing the membership values of elements.

Fuzzy hypersoft set theory represents a significant evolution of traditional fuzzy set theory, offering a more robust framework for handling uncertainty in complex systems. The adaptation of De Morgan's Laws within this framework ensures logical consistency in multi-parameter decision-making processes. This paper systematically explores these adaptations, drawing on the research of Taha Öztürk and Adem Yolcu. Additionally, it presents mathematical proofs related to inclusion and equality, further enhancing the understanding of fuzzy hypersoft sets and their applications.

2. Preliminaries

Key definitions and foundational results necessary for subsequent discussions are presented in this section.

Definition 2.1 [4] Let X represent a non-empty set, and let I denote the interval$[0,1]$.

A **Fuzzy set A** in X is expressed as:

$$A = \{(x, \mu_A(x)) : x \in X\}$$

Here, $\mu_A : X \to I$ is a membership function, often referred to as the characteristic function of the fuzzy set.

For each $x \in X$, the value $\mu_A(x)$ indicates the degree to which x belongs to the fuzzy set A. The set of *all* fuzzy subsets of X is denoted as I^X.

Definition 2.2 [2] Let X be a set, E a set of parameters related to$X$, $A \subseteq E$ and $P(X)$ the power set of X.

A pair (F, E) is called a soft set on X if F is function from E to $P(X)$.

A soft set is identified as a set of ordered pairs:

$$(F, E) = \{(e, F(e)) : e \in E, F(e) \in P(X)\}.$$

Definition 2.3 [3] Let X be the universal set and I^X be family of all fuzzy set over X. The Cartesian product of the specified parameters $\ddot{e}_1, \ddot{e}_2, \dots, \ddot{e}_n :\ n \geq 1$ corresponding to the discrete attributes $E_1, E_2, \dots, E_n$, Respectively, is denoted as $E_1 \times E_2 \times \dots \times E_n$ and represented by the symbol Σ. Given $A_i, B_i \subseteq E_i$ for $i = 1,2, \dots, n$, We simplify the notation as follows: Γ represents $A_1 \times A_2 \times \dots \times A_n$ and B represents $B_1 \times B_2 \times \dots \times B_n$. We will use α to denote any element belonging to Γ or the set B.

A pair $(\ominus, \Gamma)$ is called Fuzzy hypersoft set on X if $\ominus$ is function from Γ to I^X, i.e.

$$\ominus : \Gamma \to I^X$$

It is represented as follows:

$$(\ominus, \Gamma) = \left\{\left(\alpha, (x, \mu_{\ominus(\alpha)}(x)\right) : \mu_{\ominus(\alpha)}(x) \in [0,1], x \in X, \alpha \in \Gamma\right\}$$

Definition 2.4 [3] The complement of fuzzy hypersoft set $(\ominus, \Gamma)$ is denoted $(\ominus, \Gamma)^c$ and defined as

$$(\ominus, \Gamma)^c = (\ominus^c, \Gamma),$$

where $\ominus^c(\alpha)$ is complement of the set $\ominus(\alpha)$, for $\alpha \in \Gamma$.

Definition 2.5 [3] Let $(\ominus_1, \Gamma), (\ominus_2, B)$ be fuzzy hypersoft set on X.

1. The union of the two sets $(\ominus_1, \Gamma)$ and $(\ominus_2, B)$ denoted by $\tilde{\cup}$,

Is defined as:

$$(\ominus_1, \Gamma)\ \tilde{\cup}\ (\ominus_2, \text{B}) = (H, C)$$

Where:

$C_i = \Gamma_i \cup B_i$, $i = 1,2, \dots, n$,
and for every $\alpha \in C$:

$$H(\alpha) = \begin{cases} \ominus_1(\alpha) & if \quad \alpha \in \Gamma - B \\ \ominus_2(\alpha) \; if \; \alpha \in B - \Gamma \\ \max\{\ominus_1(\alpha), \ominus_2(\alpha)\} \, if \alpha \in \Gamma \cap B \end{cases}$$

2. The intersection of the two sets $(\ominus_1, \Gamma)$ and $(\ominus_2, B)$ denoted by $\tilde{\cap}$, is defined as:

$$\tilde{\cap}\,(\ominus_1, \Gamma) = (H, C) \;\; (\ominus_2, B)$$

Where:

$$C_i = \Gamma_i \cap B_i \neq \emptyset ,$$

and for every $\alpha \in C$:

$$H(\alpha) = \min\{\ominus_1(\alpha), \ominus_2(\alpha)\}.$$

3. De Morgan's Laws in Fuzzy Hypersoft Set Theory

In the study by Adem Yolcu and Taha Yasin Öztürk[3], Theorem (3.18) asserts the validity of De Morgan's laws within the framework of fuzzy hypersoft set theory under equality conditions. However, a detailed examination of the theorem reveals that these laws are not fully satisfied under equality; rather, only inclusion holds true, as demonstrated in the following theorem.

Theorem 3.1 Let $(\ominus_1, \Gamma), (\ominus_2, B)$ be fuzzy hypersoft set on X.

I. $((\ominus_1, \Gamma_1) \;\tilde{\cup}\; (\ominus_2, \Gamma_2))^c \;\tilde{\supseteq}\; (\ominus_1, \Gamma_1)^c \;\tilde{\cap}\; (\ominus_2, \Gamma_2)^c$

II. $((\ominus_1, \Gamma_1) \;\tilde{\cap}\; (\ominus_2, \Gamma_2))^c \;\tilde{\subseteq}\; (\ominus_1, \Gamma_1)^c \;\tilde{\cup}\; (\ominus_2, \Gamma_2)^c$

Proof:

Suppose that $(\ominus_1, \Gamma_1) \;\tilde{\cup}\; (\ominus_2, \Gamma_2) = (H, \Gamma_1 \cup \Gamma_2)$

Taking the complement of both sides

$$((\ominus_1, \Gamma_1) \;\tilde{\cup}\; (\ominus_2, \Gamma_2))^c = ((H, \Gamma_1 \cup \Gamma_2))^c = (H^c, \Gamma_1 \cup \Gamma_2)$$

Where:

$$H^c(\alpha) = 1 - H(\alpha), \forall\, \alpha \in \Gamma_1 \cup \Gamma_2 ,$$

$$H^c(\alpha) = \begin{cases} 1 - \ominus_1(\alpha) = {\ominus_1}^c(\alpha), & \alpha \in \Gamma_1 - \Gamma_2 \\ 1 - \ominus_2(\alpha) = {\ominus_2}^c(\alpha), & \alpha \in \Gamma_2 - \Gamma_1 \\ 1 - \max(\ominus_1(\alpha), \ominus_2(\alpha)) = {\ominus_1}^c(\alpha) \cap {\ominus_2}^c(\alpha), & \alpha \in \Gamma_1 \cap \Gamma_2. \end{cases}$$

Now, Consider

$$(\ominus_1, \Gamma_1)^c \;\tilde{\cap}\; (\ominus_2, \Gamma_2)^c = ({\ominus_1}^c, \Gamma_1) \;\tilde{\cap}\; ({\ominus_2}^c, \Gamma_2) = (K, \Gamma_1 \cap \Gamma_2)$$

Where:

$$K(\alpha) = \min\{{\ominus_1}^c, {\ominus_2}^c\} = {\ominus_1}^c(\alpha) \cap {\ominus_2}^c(\alpha) , \quad \forall\, \alpha \in \Gamma_1 \cap \Gamma_2$$

From the previous, we observe that K Subset H^c:

$$(\ominus_1, \Gamma_1)^c \;\tilde{\cap}\; (\ominus_2, \Gamma_2)^c \;\tilde{\subseteq}\; \left((\ominus_1, \Gamma_1) \;\tilde{\cup}\; (\ominus_2, \Gamma_2)\right)^c .$$

The second part is similar. □

Example 1: Let $X = \{x_1, x_2, x_3, x_4\}$ and the set of attributes be defined as follows:

$$E_1 = \{a_{11}, a_{12}\} \quad E_2 = \{a_{21}, a_{22}\} \; E_3 = \{a_{31}, a_{32}\}$$

Suppose that

$$A_1 = \{a_{11}, a_{12}\} \quad A_2 = \{a_{21}, a_{22}\} \; A_3 = \{a_{31}\}$$

$$B_1 = \{a_{11}\} \quad B_2 = \{a_{21}, a_{22}\} \; B_3 = \{a_{31}, a_{32}\}$$

It is clear that

$$A_i, B_i \subseteq E_i \;\; for\ each\ i = 1,2,3, \dots, n$$

Let us assume that $(\ominus_2, B)$،$(\ominus_1, \Gamma)$ are fuzzy hypersoft set defined as follows:

$$(\ominus_1, \Gamma) = \{\left((a_{11}, a_{21}, a_{31}), \{(x_1, 0.5), (x_2, 0.7)\}\right), \left((a_{11}, a_{22}, a_{31}), \{(x_2, 0.3)\}\right), \left((a_{12}, a_{21}, a_{31}), \{(x_3, 0.8), (x_4, 0.9)\}\right), \left((a_{12}, a_{22}, a_{31}), \{(x_1, 0.5), (x_4, 0.4)\}\right)\}$$

And

$$(\ominus_2, B) = \{\left((a_{11}, a_{21}, a_{31}), \{(x_2, 0.2), (x_3, 0.9)\}\right), \left((a_{11}, a_{22}, a_{31}), \{(x_2, 0.6)\}\right), ((a_{11}, a_{21}, a_{32}), \{(x_1, 0.4), (x_4, 0.7)\}), ((a_{11}, a_{22}, a_{32}), \{(x_3, 0.2), (x_4, 0.8)\})\}$$

$$\therefore ((\ominus_1, \Gamma) \;\tilde{\cup}\; (\ominus_2, B))^c = \{\left((a_{11}, a_{21}, a_{31}), \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.1), (x_4, 1)\}\right), \left((a_{11}, a_{22}, a_{31}), \{(x_1, 1), (x_2, 0.4), (x_3, 1), (x_4, 1)\}\right), \left((a_{12}, a_{21}, a_{31}), \{(x_1, 1), (x_2, 1), (x_3, 0.2), (x_4, 0.1)\}\right), \left((a_{12}, a_{22}, a_{31}), \{(x_1, 0.5), (x_2, 1), (x_3, 1)(x_4, 0.6)\}\right), \left((a_{11}, a_{21}, a_{32}), \{(x_1, 0.6), (x_2, 1), (x_3, 1), (x_4, 0.3)\}\right), \left((a_{11}, a_{22}, a_{32}), \{(x_1, 1), (x_2, 1), (x_3, 0.8), (x_4, 0.2)\}\right)\}$$

And

$$(\ominus_1, \Gamma)^c = \{\left((a_{11}, a_{21}, a_{31}), \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.1), (x_4, 1)\}\right), \left((a_{11}, a_{22}, a_{31}), \{(x_1, 1), (x_2, 0.7), (x_3, 1), (x_4, 1)\}\right), \left((a_{12}, a_{21}, a_{31}), \{(x_1, 1), (x_2, 1), (x_3, 0.2), (x_4, 0.1)\}\right), \left((a_{12}, a_{22}, a_{31}), \{(x_1, 0.5), (x_2, 1), (x_3, 1)(x_4, 0.6)\}\right)\}$$

$$(\ominus_2, \mathrm{B})^c = \{\left((a_{11}, a_{21}, a_{31}), \{(x_1, 1), (x_2, 0.8), (x_3, 0.1), (x_4, 1)\}\right), \left((a_{11}, a_{22}, a_{31}), \{(x_1, 1), (x_2, 0.4), (x_3, 1), (x_4, 1)\}\right), \left((a_{11}, a_{21}, a_{32}), \{(x_1, 0.6), (x_2, 1), (x_3, 1), (x_4, 0.3)\}\right), \left((a_{11}, a_{22}, a_{32}), \{(x_1, 1), (x_2, 1), (x_3, 0.8), (x_4, 0.2)\}\right)\}$$

$$\therefore (\ominus_1, \Gamma)^c \;\tilde{\cap}\; (\ominus_2, \mathrm{B})^c = \{\left((a_{11}, a_{21}, a_{31}), \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.1), (x_4, 1)\}\right), \left((a_{11}, a_{22}, a_{31}), \{(x_1, 1), (x_2, 0.4), (x_3, 1), (x_4, 1)\}\right)\}$$

Accordingly

$$((\ominus_1, \Gamma) \;\tilde{\cup}\; (\ominus_2, B))^c \neq (\ominus_1, \Gamma)^c \;\tilde{\cap}\; (\ominus_2, \mathrm{B})^c$$

$$((\ominus_1, \Gamma) \;\tilde{\cup}\; (\ominus_2, B))^c \;\tilde{\supseteq}\; (\ominus_1, \Gamma)^c \;\tilde{\cap}\; (\ominus_2, \mathrm{B})^c$$

Achieve equality in De Morgan's laws, we will define the restricted union and extended intersection in fuzzy hypersoft sets based on the definitions provided in [1].

Definition 3.1 Let $(\ominus_1, \Gamma), (\ominus_2, B)$ be fuzzy hypersoft set on$X$.

1) The restricted union denoted by $\cup_{\mathfrak{R}}$, is defined as:

$$(\ominus_1, \Gamma_1) \cup_{\mathfrak{R}} (\ominus_2, \Gamma_2) = (H, C)$$

Were

$$C = \Gamma_1 \cap \Gamma_2 \neq \emptyset, \forall\, \alpha \in C \;\; , H(\alpha) = \max\{\ominus_1(\alpha) \cup \ominus_2(\alpha)\}$$

2) The extended intersection denoted by $\sqcap_{\varepsilon}$, is defined as:

$$(\ominus_1, \Gamma_1) \sqcap_{\varepsilon} (\ominus_2, \Gamma_2) = (H, C)$$

Were

$$C = \Gamma_1 \cup \Gamma_2 \quad , \quad \forall\, \alpha \in C\, ,$$

$$H(\alpha) = \begin{cases} \ominus_1\,(\alpha), & \alpha \in \Gamma_1 - \Gamma_2 \\ \ominus_2\,(\alpha), & \alpha \in \Gamma_2 - \Gamma_1 \\ \min\{\ominus_1\,(\alpha), \ominus_2\,(\alpha)\}, & \alpha \in \Gamma_1 \cap \Gamma_2. \end{cases}$$

Theorem 3.2:

i. $$((\ominus_1, \Gamma_1)\, \tilde{\cap}\, (\ominus_2, \Gamma_2))^c = (\ominus_1, \Gamma_1)^c \cup_{\Re} (\ominus_2, \Gamma_2)^c$$
$$((\ominus_1, \Gamma_1) \cup_{\Re} (\ominus_2, \Gamma_2))^c = (\ominus_1, \Gamma_1)^c\, \tilde{\cap}\, (\ominus_2, \Gamma_2)^c$$

ii. $$((\ominus_1, \Gamma_1)\, \tilde{\cup}\, (\ominus_2, \Gamma_2))^c = (\ominus_1, \Gamma_1)^c \sqcap_{\varepsilon} (\ominus_2, \Gamma_2)^c$$
$$((\ominus_1, \Gamma_1) \sqcap_{\varepsilon} (\ominus_2, \Gamma_2))^c = (\ominus_1, \Gamma_1)^c\, \tilde{\cup}\, (\ominus_2, \Gamma_2)^c$$

Proof: Directly by applying the laws.

5. Conclusion

This study examined De Morgan's laws in the context of fuzzy hypersoft sets. The results confirmed that these laws hold only under inclusion, rather than strict equality. By redefining union and intersection operations, we achieved a formulation that satisfies equality conditions. These findings contribute to both theoretical advancements and practical applications of fuzzy hypersoft set theory.

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